

HOMOLOGICAL FINITENESS PROPERTIES OF FIBRE PRODUCTS

DESSISLAVA H. KOCHLOUKOVA, FRANCISMAR FERREIRA LIMA

ABSTRACT. We study the homological finiteness property FP_n of fibre products. A homological version of the n -($n+1$)-($n+2$) Conjecture is suggested and solved in some cases. Though the Homological 1-2-3 Conjecture is still open we prove a homological version of the Virtual Surjection Conjecture in the case of virtual surjection on pairs.

1. INTRODUCTION

In this paper we study homological finiteness properties FP_n of the fibre P of two epimorphisms of groups $f_1 : G_1 \rightarrow Q$ and $f_2 : G_2 \rightarrow Q$. In [15] Kuckuck studied the homotopical finiteness property F_n of the fibre P . The homotopical type F_n was defined by Wall in [22]. We recall that a group G is of type F_n if there is $K(G, 1)$ -complex with finite n -skeleton. For $n \geq 2$ a group G has a homotopical type F_n if and only if it is finitely presented and has homological type FP_n . The latest means there is a projective resolution of the trivial $\mathbb{Z}G$ -module \mathbb{Z} with finitely generated projectives in all dimensions $\leq n$, for more details and properties on the homological property FP_n we refer the reader to the Bieri book [3].

By definition, for epimorphisms of groups $f_1 : G_1 \rightarrow Q$ and $f_2 : G_2 \rightarrow Q$, the fibre product of f_1 and f_2 is

$$P = \{(g, h) | f_1(g) = f_2(h)\} \subseteq G_1 \times G_2.$$

Alternatively we say that P is the fibre product associated to the short exact sequences $Ker(f_1) \hookrightarrow G_1 \xrightarrow{f_1} Q$ and $Ker(f_2) \hookrightarrow G_2 \xrightarrow{f_2} Q$. In the case when both G_1, G_2 are finitely presented, Q is of homotopical type F_3 and one of $Ker(f_1)$ and $Ker(f_2)$ is finitely generated Bridson, Howie, Miller and Short showed in [6] that P is finitely presented. This result is called the 1-2-3 Theorem or sometimes the Asymmetric 1-2-3 Theorem. A symmetric version when $f_1 = f_2$ was proved earlier by Baumslag, Bridson, Miller and Short in [1]. Some results on finite presentability of twisted fibre products were established by Martínez-Pérez in [17] and involved the use of the Bieri-Strebel-Neumann Σ -invariant.

In [15] Kuckuck suggested

The n -($n+1$)-($n+2$) Conjecture *Let $N_1 \rightarrow G_1 \rightarrow Q$ and $N_2 \rightarrow G_2 \rightarrow Q$ be short exact sequences, where Q is of type F_{n+2} , G_1 and G_2 are groups of type F_{n+1} and N_1 is of type F_n . Then the fibre product P is of type F_{n+1} .*

In this paper we discuss a homological version of this conjecture.

The Homological n -($n+1$)-($n+2$) Conjecture *Let $N_1 \rightarrow G_1 \rightarrow Q$ and $N_2 \rightarrow G_2 \rightarrow Q$ be short exact sequences, where Q is of type FP_{n+2} , G_1 and G_2 are*

groups of type FP_{n+1} and N_1 is of type FP_n . Then the fibre product P is of type FP_{n+1} .

One of the main results in [15] is the technical [15, Prop. 4.3] and it is proved there by purely topological methods using stacks of complexes and the Borel construction. Our first result, Theorem A, is a homological version of [15, Prop. 4.3]. We prove Theorem A by purely algebraic means (spectral sequences) and observe that the original proof in [15] cannot be translated in homological language i.e. the fact that the groups are finitely presented was essentially used in [15].

Theorem A *Let $n \geq 1$ be a natural number, $A \hookrightarrow B \twoheadrightarrow C$ a short exact sequence of groups with A of type FP_n and C of type FP_{n+1} . Assume there is another short exact sequence of groups $A \hookrightarrow B_0 \twoheadrightarrow C_0$ with B_0 of type FP_{n+1} and that there is a group homomorphism $\theta : B_0 \rightarrow B$ such that $\theta|_A = id_A$, i.e. there is a commutative diagram of homomorphisms of groups*

$$\begin{array}{ccccc} A & \hookrightarrow & B_0 & \twoheadrightarrow & C_0 \\ id_A \downarrow & & \theta \downarrow & & \downarrow \nu \\ A & \hookrightarrow & B & \twoheadrightarrow & C \end{array}$$

Then B is of type FP_{n+1} .

The homotopical version of Theorem A, [15, Prop. 4.3], was used in [15] to prove several results about the n -($n+1$)-($n+2$) Conjecture. Here we adopt the same approach and following the recipe suggested in [15] we deduce from Theorem A several results about the Homological n -($n+1$)-($n+2$) Conjecture.

Theorem B *The Homological n -($n+1$)-($n+2$) Theorem holds if the second sequence splits.*

Theorem C *If the Homological n -($n+1$)-($n+2$) Conjecture holds whenever G_2 is a finitely generated free group then it holds in general.*

The proof of the following result uses some properties of the homological Σ -invariants defined by Bieri and Renz in [5]. In Section 2 we will revise the properties of the homological Σ -invariants that will be needed later.

Theorem D *Let $n \geq 1$ be a natural number, $N_1 \hookrightarrow G_1 \xrightarrow{\pi_1} Q$, $N_2 \hookrightarrow G_2 \xrightarrow{\pi_2} Q$ be short exact sequences of groups, where G_1, G_2 are of type FP_{n+1} , Q is virtually abelian, N_1 is of type FP_k and N_2 is of type FP_l for some $k, l \geq 0$ with $k + l \geq n$. Then the fiber product P of π_1 and π_2 is of type FP_{n+1} .*

Though the general case of the Homological 1-2-3 Conjecture is still open, we solve it in the case when Q is finitely presented.

Theorem E *The Homological 1-2-3 Conjecture holds if Q is finitely presented.*

Our interest in the homological finiteness properties of fibre products stems from our interest in the homological finiteness type of subgroups of direct products of groups. Some results about the homotopical type F_n were conjectured in the case of subdirect products of non-abelian limit groups by Dison in [9, section 12.5] and in the case of some special subdirect products of groups of type FP_∞ by Kochloukova in [12]. Limit groups were defined by Sela and studied by Kharlampovich and Myasnikov under the name fully residually free groups. The class of limit groups played an important role in the solution of the Tarski problem in [11] and [21].

The interest in the study of homological and homotopical properties of subdirect products derives from the fact that every finitely generated residually free group embeds as a subgroup of a direct product of finitely many limit groups [2].

The homotopical type of subdirect products of groups was conjectured in [15], where Kuckuck stated the following form of the Virtual Surjection Theorem.

The Virtual Surjection Conjecture *Let $n \geq 2$ be a natural number, G_1, \dots, G_k be groups of homotopical type F_n , where $n \leq k$ and $P \subseteq G_1 \times \dots \times G_k$ be a subgroup that virtually surjects on every n factors i.e. for every $1 \leq i_1 < \dots < i_n \leq k$ the image of P under the canonical projection $P \rightarrow G_{i_1} \times G_{i_2} \times \dots \times G_{i_n}$ has finite index. Then P is of type F_n .*

In [6] Bridson, Howie, Miller and Short showed that the Virtual Surjection Conjecture holds for $n = 2$ and this was deduced as a corollary of the 1-2-3 Theorem. This was later generalised in [15], where Kuckuck proved that if the $(n-1)$ - n - $(n+1)$ Conjecture holds when Q is virtually nilpotent then the Virtual Surjection Conjecture holds in general. In [7] Bridson, Howie, Miller and Short proved that if P is a finitely presented subdirect product of non-abelian limit groups G_1, \dots, G_k such that $P \cap G_i \neq \emptyset$ for every $1 \leq i \leq k$ then P virtually surjects on pairs. Later in [12], Kochloukova showed that if furthermore P is of type FP_n for some $n \leq k$ then P virtually surjects on every n factors. In this paper we suggest the following homological version of the Virtual Surjection Conjecture.

The Homological Virtual Surjection Conjecture *Let $n \geq 2$ be a natural number and G_1, \dots, G_k be groups of homological type FP_n , where $n \leq k$ and $P \subseteq G_1 \times \dots \times G_k$ be a subgroup that virtually surjects on every n factors. Then P is of type FP_n .*

The first part of Theorem F is a homological version of [15, Thm. 3.10]. The second part of Theorem F follows from the first part, Theorem E and the fact that every virtually nilpotent group is finitely presented.

Theorem F *If the Homological $(n-1)$ - n - $(n+1)$ Conjecture holds for Q virtually nilpotent then the Homological Virtual Surjection Conjecture holds in general. In particular, the Homological Virtual Surjection Conjecture holds for $n = 2$ i.e. for groups that virtually surject on pairs.*

Finally we note that some results on homological finiteness properties of fibre sums of Lie algebras and subdirect sums of Lie algebras were recently established by Kochloukova and Martínez-Pérez in [13]. Though in the Lie algebra case there are no homotopic methods available, a version of the 1-2-3 Theorem for Lie algebras was proved in [13].

2. PRELIMINARIES ON THE HOMOLOGICAL TYPE FP_m AND HOMOLOGICAL Σ -INVARIANTS

2.1. Preliminaries on the homological type FP_m . If not otherwise stated the modules considered in this paper are left ones.

Definition. *Let S be an associative ring with 1. An S -module M is said to be of type FP_n if there is a projective resolution*

$$\dots \rightarrow P_i \rightarrow P_{i-1} \rightarrow \dots \rightarrow P_0 \rightarrow M \rightarrow 0$$

with P_i finitely generated for all $i \leq n$. We say that a group G is of type FP_n if the trivial $\mathbb{Z}G$ -module \mathbb{Z} is of type FP_n .

We will need later the following criterion for modules of type FP_n .

Lemma 2.1. [3, Prop. 1.2, Thm 1.3+remarks] *Let S be an associative ring with 1 and $n \geq 1$ be a natural number. The following are equivalent for S -module M :*

- 1) M is of type FP_n ;
- 2) for a direct product $\prod S$ of arbitrary many copies of S we have $\text{Tor}_k^S(\prod S, M) = 0$ for $1 \leq k \leq n-1$ and M is finitely presented as S -module;
- 3) the functor $\text{Tor}_k^S(-, M)$ commutes with arbitrary direct product for $0 \leq k \leq n-1$.

Remark We will apply the above lemma for $S = \mathbb{Z}G$, where G is a finitely generated group and for $M = \mathbb{Z}$ the trivial $\mathbb{Z}G$ -module. In this case M is automatically finitely presented as S -module.

The following result is well known and can be deduced after making appropriate modifications to the proof of [3, Prop. 2.7], which uses spectral sequences and Lemma 2.1. A detailed proof can be found in Lima's PhD thesis [16].

Proposition 2.2. *Let $A \rightarrow B \rightarrow C$ be a short exact sequence of groups.*

- a) *if both A and C are of type FP_n then B is of type FP_n ;*
- b) *if A is of type FP_n and B is of type FP_{n+1} then C is of type FP_{n+1} .*

2.2. Preliminaries on the homological Σ -invariants. For a finitely generated group G we define $S(G) = \text{Hom}(G, \mathbb{R}) \setminus \{0\} / \sim$, where for two characters $\chi_1, \chi_2 \in \text{Hom}(G, \mathbb{R}) \setminus \{0\}$ we have $\chi_1 \sim \chi_2$ if there is a positive real number r such that $\chi_1 = r\chi_2$. We write $[\chi]$ for the equivalence class of χ with respect to \sim . Thus

$$S(G) \simeq S^{n-1},$$

where n is the torsion-free rank of the abelianization of G . The n -dimensional Bieri-Renz Σ -invariant is defined by

$$\Sigma^n(G, \mathbb{Z}) = \{[\chi] \mid \mathbb{Z} \text{ is of type } FP_n \text{ as } \mathbb{Z}G_\chi\text{-module}\},$$

where G_χ is the monoid $\{g \in G \mid \chi(g) \geq 0\}$. The following results will be used later in the paper.

Theorem 2.3. [5, Theorem 5.1] *Let $n \geq 1$ be a natural number, G a group of type FP_n and $N \subseteq G$ a normal subgroup such that G/N is abelian. Then, N is of type FP_n if and only if*

$$S(G, N) \subseteq \Sigma^n(G, \mathbb{Z}).$$

The following result was published in [10]. As stated in [10] the result was proved (unpublished) by Meinert and generalizes ideas from [19].

Theorem 2.4. [10, Lemma 9.1] *Let G_1, G_2 be groups of type FP_n with $n \geq 1$ and $\chi : G_1 \times G_2 \rightarrow \mathbb{R}$ be a character such that $\chi|_{(G_1 \times 1)} \neq 0$ and $\chi|_{(1 \times G_2)} \neq 0$. If $[\chi|_{(G_1 \times 1)}] \in \Sigma^k(G_1, \mathbb{Z})$ and $[\chi|_{(1 \times G_2)}] \in \Sigma^l(G_2, \mathbb{Z})$ for some $k, l \geq 0$ with $k + l < n$, then $[\chi] \in \Sigma^{k+l+1}(G_1 \times G_2, \mathbb{Z})$.*

The above result was one of the reasons to believe in special formula for calculating Σ -invariants of direct product of groups, known as the direct product conjecture for sigma invariants. As shown by Bieri and Geoghegan in [4] this conjecture holds for the invariants $\Sigma^n(G, R)$, where R is a field and $\Sigma^n(G, R) = \{[\chi] \mid$

R is of type FP_n as RG_χ -module}. It turned out the conjecture is wrong in general, Schütz proved in [20] that the conjecture does not hold for $\Sigma^n(G, \mathbb{Z})$ when $n \geq 4$.

3. HOMOLOGICAL VERSION OF A RESULT OF KUCKUCK

In this section we prove a homological version of [15, Prop. 4.3] with algebraic methods. The starting point is the following result that is based on the classical Lyndon-Hoschild-Serre spectral sequence.

Theorem 3.1. *Let I be an index set, $n \geq 1$ a natural number and $A \hookrightarrow B \twoheadrightarrow C$ a short exact sequence of groups. Furthermore we assume that A is of type FP_n and B is of type FP_{n+1} , M is a free $\mathbb{Z}B$ -module and consider the LHS spectral sequence*

$$E_{p,q}^2 = H_p(C, H_q(A, \prod_{\alpha \in I} M_\alpha))$$

converging to $H_{p+q}(B, \prod_{\alpha \in I} M_\alpha)$, where $M_\alpha = M$ for $\alpha \in I$. Then

$$E_{n+1,0}^{n+1} = E_{n+1,0}^2 = H_{n+1}(C, H_0(A, \prod_{\alpha \in I} M_\alpha)), E_{0,n}^{n+1} = E_{0,n}^2 = H_0(C, H_n(A, \prod_{\alpha \in I} M_\alpha))$$

and the differential

$$d_{n+1,0}^{n+1} : H_{n+1}(C, H_0(A, \prod_{\alpha \in I} M_\alpha)) \longrightarrow H_0(C, H_n(A, \prod_{\alpha \in I} M_\alpha))$$

is surjective.

Proof. First we assume $n = 1$. In this case $d_{2,0}^2 : E_{2,0}^2 \rightarrow E_{0,1}^2$. Since B is FP_2 by Lemma 2.1 we have $H_1(B, \prod_{\alpha \in I} M_\alpha) = \prod_{\alpha \in I} H_1(B, M_\alpha) = 0$, hence by the convergence of the LHS spectral sequence $E_{0,1}^\infty = 0$. Note that all differentials that enter and leave $E_{0,1}^j$ are zero if $j \geq 3$, so $0 = E_{0,1}^\infty = E_{0,1}^3 = E_{0,1}^2 / \text{Im}(d_{2,0}^2)$, hence $d_{2,0}^2$ is surjective.

From now on we assume that $n \geq 2$. We split the proof in several steps.

Step 1. Note that by Proposition 2.2 b) C is of type FP_{n+1} . Since A is of type FP_n , by Lemma 2.1, we have that

$$H_q(A, \prod_{\alpha \in I} M_\alpha) = \prod_{\alpha \in I} H_q(A, M_\alpha) \text{ for } 0 \leq q \leq n-1.$$

Now since $M_\alpha = M$ is a free $\mathbb{Z}B$ -module for $\alpha \in I$, we have $H_q(A, M_\alpha) = 0$ for $q \geq 1$ and

$$M \cong \bigoplus_{\beta \in J} (\mathbb{Z}B)_\beta \text{ for an index set } J,$$

where $(\mathbb{Z}B)_\beta = \mathbb{Z}B$ for $\beta \in J$. Since direct sum commutes with tensor product for all $\alpha \in I$, we have

$$H_0(A, M_\alpha) \cong \mathbb{Z} \otimes_{\mathbb{Z}A} (\bigoplus_{\beta \in J} (\mathbb{Z}B)_\beta) \cong \bigoplus_{\beta \in J} (\mathbb{Z}(B/A))_\beta \cong \bigoplus_{\beta \in J} (\mathbb{Z}C)_\beta =: \tilde{M},$$

where $(\mathbb{Z}C)_\beta = \mathbb{Z}C$ for $\beta \in J$. Thus

$$(3.1) \quad E_{p,q}^2 = \begin{cases} \mathbf{0} & \text{if } 1 \leq q \leq n-1 \\ H_p(C, \prod_{\alpha \in I} \tilde{M}_\alpha) & \text{if } q = 0 \end{cases}$$

where $\tilde{M}_\alpha = \tilde{M}$ for all $\alpha \in I$ and \tilde{M} is a free $\mathbb{Z}C$ -module. Since C is of type FP_{n+1} , by Lemma 2.1

$$(3.2) \quad H_p(C, \prod_{\alpha \in I} \tilde{M}_\alpha) \cong \prod_{\alpha \in I} H_p(C, \tilde{M}_\alpha) \text{ for } 0 \leq p \leq n$$

and since \tilde{M}_α is a free $\mathbb{Z}C$ -module, we have that

$$(3.3) \quad H_p(C, \tilde{M}_\alpha) = \mathbf{0}, \text{ for all } \alpha \in I \text{ and } p \geq 1.$$

It follows by (3.2) and (3.3) that

$$(3.4) \quad H_p(C, \prod_{\alpha \in I} \tilde{M}_\alpha) = \mathbf{0}, \text{ for } 1 \leq p \leq n.$$

Thus we obtain by (3.1), that

$$(3.5) \quad E_{p,q}^2 = \mathbf{0}, \text{ if } 1 \leq q \leq n-1, \text{ or } q = 0 \text{ and } 1 \leq p \leq n.$$

Step 2. Consider the differentials

$$E_{i,n+1-i}^i \xrightarrow{d_{i,n+1-i}^i} E_{0,n}^i \xrightarrow{d_{0,n}^i} E_{-i,n+i-1}^i = \mathbf{0},$$

where $E_{-i,n+i-1}^i = 0$ since $-i < 0$. By definition

$$(3.6) \quad E_{0,n}^{i+1} = \frac{\ker(d_{0,n}^i)}{\operatorname{im}(d_{i,n+1-i}^i)} = \frac{E_{0,n}^i}{\operatorname{im}(d_{i,n+1-i}^i)}.$$

By (3.5)

$$(3.7) \quad E_{i,n+1-i}^2 = \mathbf{0}, \text{ if } 2 \leq i \leq n.$$

On other hand for $i \geq n+2$ we have $n+1-i < 0$ and this implies that $E_{i,n+1-i}^2 = \mathbf{0}$ for $i \geq n+2$. Using this and (3.7), we conclude that $E_{i,n+1-i}^2 = 0$ if $i \neq n+1$ and $i \geq 2$. Hence

$$E_{i,n+1-i}^i = \mathbf{0} \text{ if } i \neq n+1 \text{ and } i \geq 2,$$

and this implies that

$$(3.8) \quad \operatorname{im}(d_{i,n+1-i}^i) = \mathbf{0} \text{ if } i \neq n+1 \text{ and } i \geq 2.$$

By (3.6) and (3.8) we obtain that

$$E_{0,n}^{i+1} = E_{0,n}^i, \text{ if } i \neq n+1 \text{ and } i \geq 2,$$

hence

$$(3.9) \quad E_{0,n}^2 = E_{0,n}^3 = \dots = E_{0,n}^n = E_{0,n}^{n+1} \quad \text{and} \quad E_{0,n}^{n+2} = E_{0,n}^{n+3} = \dots = E_{0,n}^\infty.$$

This implies that $d_{n+1,0}^{n+1} : E_{n+1,0}^{n+1} \rightarrow E_{0,n}^{n+1}$ has as codomain $E_{0,n}^2 = H_0(C, H_n(A, \prod_{\alpha \in I} M_\alpha))$.

Step 3. We will show that $E_{n+1,0}^{n+1} = E_{n+1,0}^2$.

Consider the differentials

$$E_{n+1+i,1-i}^i \xrightarrow{d_{n+1+i,1-i}^i} E_{n+1,0}^i \xrightarrow{d_{n+1,0}^i} E_{n+1-i,i-1}^i$$

For $i \geq 2$, we have $E_{n+1+i,1-i}^i = \mathbf{0}$, since $1-i < 0$. Then, $\text{im}(d_{n+1+i,1-i}^i) = \mathbf{0}$ and so

$$(3.10) \quad E_{n+1,0}^{i+1} = \frac{\ker(d_{n+1,0}^i)}{\text{im}(d_{n+1+i,1-i}^i)} = \ker(d_{n+1,0}^i).$$

Now, by (3.5), $E_{n+1-i,i-1}^i = \mathbf{0}$ if $1 \leq i-1 \leq n-1$. Then $\ker(d_{n+1,0}^i) = E_{n+1,0}^i$ if $2 \leq i \leq n$ and this implies $E_{n+1,0}^{i+1} = E_{n+1,0}^i$ if $2 \leq i \leq n$. Hence

$$E_{n+1,0}^2 = E_{n+1,0}^3 = \dots = E_{n+1,0}^n = E_{n+1,0}^{n+1}$$

and so

$$E_{n+1,0}^{n+1} = E_{n+1,0}^2 = H_{n+1}(C, H_0(A, \prod_{\alpha \in I} M_\alpha)).$$

Step 4. By the convergence of the LHS spectral sequence there is a filtration

$$(3.11) \quad \mathbf{0} = \Phi^{-1}H_n \subseteq \Phi^0H_n \subseteq \dots \subseteq \Phi^{n-1}H_n \subseteq \Phi^nH_n = H_n$$

such that

$$(3.12) \quad E_{p,q}^\infty \cong \Phi^pH_n / \Phi^{p-1}H_n \text{ for } p+q = n,$$

where H_n denotes $H_n(B, \prod_{\alpha \in I} M_\alpha)$. By (3.5) we deduce that $E_{p,q}^2 = \mathbf{0}$ if $p+q = n$ and $p \neq 0$. Hence

$$(3.13) \quad E_{p,q}^\infty = \mathbf{0}, \text{ if } p+q = n \text{ and } p \neq 0.$$

Using the filtration (3.11), by (3.12) and (3.13), we get

$$\mathbf{0} = \Phi^{-1}H_n \subseteq \Phi^0H_n = \dots = \Phi^{n-1}H_n = \Phi^nH_n = H_n.$$

Then

$$(3.14) \quad E_{0,n}^\infty \cong \Phi^0H_n / \Phi^{-1}H_n = H_n = H_n(B, \prod_{\alpha \in I} M_\alpha).$$

Note that by now we have used only the fact that A is FP_n and C is FP_{n+1} but we have **not used** that B is of type FP_{n+1} . Since B is of type FP_{n+1} , by Lemma 2.1 and since for all $\alpha \in I$, $M_\alpha = M$ is a free $\mathbb{Z}B$ -module we obtain that

$$H_n(B, \prod_{\alpha \in I} M_\alpha) \cong \prod_{\alpha \in I} H_n(B, M_\alpha) = \prod_{\alpha \in I} 0 = \mathbf{0}$$

and so

$$(3.15) \quad E_{0,n}^\infty = \mathbf{0}$$

By (3.9) and (3.15) we have

$$(3.16) \quad E_{0,n}^{n+2} = \mathbf{0}.$$

Consider the differentials

$$E_{n+1,0}^{n+1} \xrightarrow{d_{n+1,0}^{n+1}} E_{0,n}^{n+1} \xrightarrow{d_{0,n}^{n+1}} E_{-n-1,2n}^{n+1} = \mathbf{0}$$

and note that

$$E_{0,n}^{n+2} = \frac{\ker(d_{0,n}^{n+1})}{\text{im}(d_{n+1,0}^{n+1})} = \frac{E_{0,n}^{n+1}}{\text{im}(d_{n+1,0}^{n+1})}.$$

Then by (3.16) we obtain that $d_{n+1,0}^{n+1}$ is surjective. □

Proposition 3.2. *Let $n \geq 1$ be a natural number, B a group with a normal subgroup A such that A is of type FP_n and $C = B/A$ is of type FP_{n+1} . Then B is of type FP_{n+1} if and only if for any direct product the map*

$$d_{n+1,0}^{n+1} : H_{n+1}(C, H_0(A, \prod \mathbb{Z}B)) \longrightarrow H_0(C, H_n(A, \prod \mathbb{Z}B))$$

is surjective, where $d_{n+1,0}^{n+1}$ is the differential from LHS spectral sequence $E_{p,q}^2 = H_p(C, H_q(A, \prod \mathbb{Z}B))$ converging to $H_{p+q}(B, \prod \mathbb{Z}B)$.

Proof. Note that one of the direction is precisely Theorem 3.1. Assume from now on that the differential $d_{n+1,0}^{n+1}$ is surjective. We apply the notations of the proof of Theorem 3.1 with $M = \mathbb{Z}B$. Note that in the proof of (3.14) we have not used that B is of type FP_{n+1} . Then we can apply (3.14) for $M = \mathbb{Z}B$ and deduce by Lemma 2.1 that

$$B \text{ is of type } FP_{n+1} \text{ if and only if } E_{0,n}^\infty = 0.$$

Note that in (3.9) of the proof of Theorem 3.1 we have not used that B is FP_{n+1} . By (3.9) we have

$$E_{0,n}^2 = E_{0,n}^{n+1}, E_{0,n}^{n+2} = E_{0,n}^\infty.$$

Thus

$$B \text{ is of type } FP_{n+1} \text{ if and only if } 0 = E_{0,n}^{n+2}.$$

Since $d_{0,n}^{n+1} : E_{0,n}^{n+1} \rightarrow E_{-n-1,2n}^{n+1} = 0$ is the zero map, $E_{0,n}^{n+2} = \ker(d_{0,n}^{n+1})/\text{im}(d_{n+1,0}^{n+1}) = E_{0,n}^{n+1}/\text{im}(d_{n+1,0}^{n+1})$ is the cokernel of $d_{n+1,0}^{n+1}$. Thus

$$0 = E_{0,n}^{n+2} \text{ if and only if } d_{n+1,0}^{n+1} \text{ is surjective.}$$

□

Corollary 3.3. *Let $n \geq 1$ be a natural number, $A \rightarrow B \rightarrow C$ be a short exact sequence of groups such that A is of type FP_n and C is of type FP_{n+1} . Then*

- a) if $H_0(C, H_n(A, \prod \mathbb{Z}B)) = 0$ for any direct product $\prod \mathbb{Z}B$ then B is of type FP_{n+1} ;*
- b) if C is of type FP_{n+2} then*

$$B \text{ is of type } FP_{n+1} \text{ if and only if } H_0(C, H_n(A, \prod \mathbb{Z}B)) = 0$$

for any direct product $\prod \mathbb{Z}B$.

Proof. Part a) follows directly from Proposition 3.2, since in this case the co-domain of $d_{n+1,0}^{n+1}$ is 0, so $d_{n+1,0}^{n+1}$ is surjective.

To prove part b) note that if C is of type FP_{n+2} then $H_{n+1}(C, \prod \mathbb{Z}C) = \prod H_{n+1}(C, \mathbb{Z}C) = \prod 0 = 0$. Since A is finitely generated $H_0(A, -)$ commutes with direct products, hence

$$H_{n+1}(C, H_0(A, \prod \mathbb{Z}B)) \simeq H_{n+1}(C, \prod H_0(A, \mathbb{Z}B)) = H_{n+1}(C, \prod \mathbb{Z}C) = 0.$$

Thus the domain of the map $d_{n+1,0}^{n+1}$ is 0, so $d_{n+1,0}^{n+1}$ is the zero map and $d_{n+1,0}^{n+1}$ is surjective if and only if $H_0(C, H_n(A, \prod \mathbb{Z}B)) = 0$. Finally by Proposition 3.2 B is FP_{n+1} if and only if $d_{n+1,0}^{n+1}$ is surjective. □

Below we restate and prove Theorem A.

Theorem 3.4. *Let $n \geq 1$ be a natural number, $A \hookrightarrow B \twoheadrightarrow C$ a short exact sequence of groups with A of type FP_n and C of type FP_{n+1} . Assume there is another short exact sequence of groups $A \hookrightarrow B_0 \twoheadrightarrow C_0$ with B_0 of type FP_{n+1} and that there is a group homomorphism $\theta : B_0 \rightarrow B$ such that $\theta|_A = id_A$, i.e. there is a commutative diagram of homomorphisms of groups*

$$\begin{array}{ccccc} A & \hookrightarrow & B_0 & \twoheadrightarrow & C_0 \\ id_A \downarrow & & \theta \downarrow & & \downarrow \nu \\ A & \hookrightarrow & B & \twoheadrightarrow & C \end{array}$$

Then B is of type FP_{n+1} .

Proof. We break the proof in several steps.

Step 1. Consider the LHS spectral sequence

$$E_{p,q}^2 = H_p(C_0, H_q(A, \mathbb{Z}B)) \rightrightarrows_p H_{p+q}(B_0, \mathbb{Z}B)$$

i.e. this is the standard Lyndon-Hoschild-Serre spectral sequence applied for the short exact sequence $A \rightarrow B_0 \rightarrow C_0$ and the $\mathbb{Z}B_0$ -module $\mathbb{Z}B$, where we view $\mathbb{Z}B$ as a $\mathbb{Z}B_0$ -module via θ . Note that $H_q(A, \mathbb{Z}B) = \mathbf{0}$ for $q \geq 1$ since $\mathbb{Z}B$ is a free $\mathbb{Z}A$ -module and

$$(3.17) \quad H_0(A, \mathbb{Z}B) \cong \mathbb{Z} \otimes_{\mathbb{Z}A} \mathbb{Z}B \cong \mathbb{Z}(B/A) \cong \mathbb{Z}C.$$

It follows that

$$(3.18) \quad E_{p,q}^2 = \begin{cases} \mathbf{0} & \text{if } q \geq 1 \\ H_p(C_0, \mathbb{Z}C) & \text{if } q = 0, \end{cases}$$

hence the spectral sequence collapses and $E_{n,0}^\infty = E_{n,0}^2 = H_n(C_0, \mathbb{Z}C)$ for every $n \geq 0$. By the convergence of the LHS spectral sequence there is a filtration

$$\mathbf{0} = \Phi^{-1}H_n \subseteq \Phi^0H_n \subseteq \dots \subseteq \Phi^{n-1}H_n \subseteq \Phi^nH_n = H_n$$

such that

$$E_{p,q}^\infty \cong \Phi^pH_n / \Phi^{p-1}H_n \text{ for } p+q = n,$$

where H_n denotes $H_n(B_0, \mathbb{Z}B)$. Thus

$$(3.19) \quad \mathbf{0} = \Phi^{-1}H_n = \Phi^0H_n = \dots = \Phi^{n-1}H_n \subseteq \Phi^nH_n = H_n$$

and so the homomorphism

$$(3.20) \quad \varphi : H_n = H_n(B_0, \mathbb{Z}B) \rightarrow E_{n,0}^\infty \simeq H_n(C_0, \mathbb{Z}C) \text{ is an isomorphism for } n \geq 0,$$

where φ is induced by $\pi_0 : B_0 \twoheadrightarrow C_0$ and by the homomorphism $\pi_\# : \mathbb{Z}B \twoheadrightarrow \mathbb{Z}C$ that itself is induced by π .

Step 2. Let I be an index set. Consider the LHS spectral sequence

$$\mathcal{E}_{p,q}^2 = H_p(C_0, H_q(A, \prod_{\alpha \in I} (\mathbb{Z}B)_\alpha)) \rightrightarrows_p H_{p+q}(B_0, \prod_{\alpha \in I} (\mathbb{Z}B)_\alpha),$$

associated to the short exact sequence of groups $A \rightarrow B_0 \rightarrow C_0$, where $(\mathbb{Z}B)_\alpha = \mathbb{Z}B$ for $\alpha \in I$ and $\mathbb{Z}B$ is $\mathbb{Z}B_0$ -module via θ . Since A is of type FP_n by Lemma 2.1, it follows that

$$(3.21) \quad H_q(A, \prod_{\alpha \in I} (\mathbb{Z}B)_\alpha) = \prod_{\alpha \in I} H_q(A, (\mathbb{Z}B)_\alpha) \text{ for } 0 \leq q \leq n-1.$$

Furthermore $H_q(A, (\mathbb{Z}B)_\alpha) = H_q(A, \mathbb{Z}B) = \mathbf{0}$ for $q \geq 1$, since $\mathbb{Z}B$ is a free $\mathbb{Z}A$ -module. Then

$$(3.22) \quad \mathcal{E}_{p,q}^2 = \mathbf{0} \text{ if } 1 \leq q \leq n-1.$$

Then since $\mathcal{E}_{p,q}^\infty$ is a subquotient of $\mathcal{E}_{p,q}^2$ we obtain that

$$(3.23) \quad \mathcal{E}_{p,q}^\infty = \mathbf{0} \text{ if } 1 \leq q \leq n-1.$$

Observe that C_0 is of type FP_{n+1} by Proposition 2.2, hence by Lemma 2.1 the functor $H_n(C_0, -)$ commutes with direct products. Since A is finitely generated the functor $H_0(A, -)$ commutes with direct products. This together with (3.17) implies the isomorphisms

$$(3.24) \quad \begin{aligned} \mathcal{E}_{n,0}^2 &= H_n(C_0, H_0(A, \prod_{\alpha \in I} (\mathbb{Z}B)_\alpha)) \cong H_n(C_0, \prod_{\alpha \in I} H_0(A, (\mathbb{Z}B)_\alpha)) \cong \\ &\prod_{\alpha \in I} H_n(C_0, H_0(A, (\mathbb{Z}B)_\alpha)) \cong \prod_{\alpha \in I} H_n(C_0, (\mathbb{Z}C)_\alpha), \end{aligned}$$

where $(\mathbb{Z}C)_\alpha = \mathbb{Z}C$ and $(\mathbb{Z}B)_\alpha = \mathbb{Z}B$, for every $\alpha \in I$.

Step 3. Consider the differentials

$$\mathcal{E}_{n+i,1-i}^i \xrightarrow{\delta_{n+i,1-i}^i} \mathcal{E}_{n,0}^i \xrightarrow{\delta_{n,0}^i} \mathcal{E}_{n-i,i-1}^i.$$

Then $\mathcal{E}_{n+i,1-i}^i = \mathbf{0}$, since $1-i < 0$, and furthermore $\mathcal{E}_{n-i,i-1}^i = \mathbf{0}$ for $2 \leq i \leq n$ by (3.22). Note that $\mathcal{E}_{n-i,i-1}^i = \mathbf{0}$ for $i > n$, since in this case $n-i < 0$. Thus

$$\mathcal{E}_{n,0}^{i+1} = \frac{\ker(\delta_{n,0}^i)}{\text{im}(\delta_{n+i,1-i}^i)} = \mathcal{E}_{n,0}^i \text{ for every } i \geq 2.$$

This together with (3.24) implies

$$(3.25) \quad \mathcal{E}_{n,0}^\infty = \mathcal{E}_{n,0}^2 \simeq H_n(C_0, \prod_{\alpha \in I} (\mathbb{Z}C)_\alpha).$$

By the convergence of the spectral sequence there is a filtration

$$(3.26) \quad \mathbf{0} = \Lambda^{-1}H_n \subseteq \Lambda^0H_n \subseteq \dots \subseteq \Lambda^{n-1}H_n \subseteq \Lambda^nH_n = H_n,$$

such that

$$(3.27) \quad \mathcal{E}_{p,q}^\infty \cong \Lambda^pH_n / \Lambda^{p-1}H_n \text{ for } p+q=n,$$

where to simplify the notation we denote $H_n(B_0, \prod_{\alpha \in I} (\mathbb{Z}B)_\alpha)$ by H_n . By (3.23) and (3.27) we have

$$\mathbf{0} = \Lambda^{-1}H_n \subseteq \Lambda^0H_n = \dots = \Lambda^{n-1}H_n \subseteq \Lambda^nH_n = H_n = H_n(B_0, \prod_{\alpha \in I} (\mathbb{Z}B)_\alpha).$$

Thus, on one hand

$$\mathcal{E}_{0,n}^\infty \cong \Lambda^0H_n / \Lambda^{-1}H_n \cong \Lambda^0H_n = \Lambda^{n-1}H_n.$$

And on other hand

$$\mathcal{E}_{n,0}^\infty \cong \Lambda^nH_n / \Lambda^{n-1}H_n = H_n / \Lambda^{n-1}H_n \cong H_n / \mathcal{E}_{0,n}^\infty.$$

Then we have a short exact sequence of groups

$$(3.28) \quad \mathcal{E}_{0,n}^\infty \hookrightarrow H_n \xrightarrow{\widehat{\theta}} \mathcal{E}_{n,0}^\infty,$$

where the epimorphism $\widehat{\theta}$ is induced by the epimorphism $\pi_0 : B_0 \twoheadrightarrow C_0$ e $\pi_\# : \mathbb{Z}B \twoheadrightarrow \mathbb{Z}C$, where $\pi_\#$ is a ring epimorphism induced by the epimorphism of groups π .

Step 4. We claim that

$$\widehat{\theta} : H_n(B_0, \prod_{\alpha \in I} (\mathbb{Z}B)_\alpha) \rightarrow H_n(C_0, \prod_{\alpha \in I} (\mathbb{Z}C)_\alpha)$$

is an isomorphism. In fact, by (3.20), there is a group isomorphism φ that induces an isomorphism

$$\Pi\varphi : \prod_{\alpha \in I} H_n(B_0, (\mathbb{Z}B)_\alpha) \xrightarrow{\sim} \prod_{\alpha \in I} H_n(C_0, (\mathbb{Z}C)_\alpha).$$

Since B_0 and C_0 are FP_{n+1} , we have that both functors $H_n(B_0, -)$ and $H_n(C_0, -)$ commute with direct products. Then, $\Pi\varphi$ induces the homomorphism of groups $\widehat{\theta}$. Then by (3.28)

$$(3.29) \quad \mathcal{E}_{0,n}^\infty = \text{Ker}(\widehat{\theta}) = \mathbf{0}.$$

Step 5. Consider the differentials for $i \geq 2$

$$\mathcal{E}_{i,n+1-i}^i \xrightarrow{\delta_{i,n+1-i}^i} \mathcal{E}_{0,n}^i \xrightarrow{\delta_{0,n}^i} \mathcal{E}_{-i,n+i-1}^i.$$

Then by (3.22), $\mathcal{E}_{i,n+1-i}^i = \mathbf{0}$ for $2 \leq i \leq n$ and $\mathcal{E}_{i,n+1-i}^i = \mathbf{0}$ for $i \geq n+2$ since $n+1-i < 0$. Furthermore $\mathcal{E}_{-i,n+i-1}^i = 0$, since $-i < 0$. Thus we have

$$\mathcal{E}_{0,n}^{i+1} = \frac{\text{ker}(\delta_{0,n}^i)}{\text{im}(\delta_{i,n+1-i}^i)} \cong \mathcal{E}_{0,n}^i \text{ for every } i \geq 2 \text{ and } i \neq n+1.$$

Then

$$(3.30) \quad \mathcal{E}_{0,n}^2 \cong \dots \cong \mathcal{E}_{0,n}^{n+1} \quad \text{and} \quad \mathcal{E}_{0,n}^{n+2} \cong \dots \cong \mathcal{E}_{0,n}^\infty.$$

By (3.29) and (3.30) we have that

$$(3.31) \quad \mathcal{E}_{0,n}^{n+2} \cong \mathcal{E}_{0,n}^\infty = \mathbf{0}.$$

Step 6. Consider the differentials

$$\mathcal{E}_{n+1,0}^{n+1} \xrightarrow{\delta_{n+1,0}^{n+1}} \mathcal{E}_{0,n}^{n+1} \xrightarrow{\delta_{0,n}^{n+1}} \mathcal{E}_{-n-1,2n}^{n+1}.$$

Note that $\mathcal{E}_{-n-1,2n}^{n+1} = \mathbf{0}$, since $-n-1 < 0$, then $\text{ker}(\delta_{0,n}^{n+1}) = \mathcal{E}_{0,n}^{n+1}$. This together with (3.31) implies

$$\mathbf{0} = \mathcal{E}_{0,n}^{n+2} := \frac{\text{ker}(\delta_{0,n}^{n+1})}{\text{im}(\delta_{n+1,0}^{n+1})} = \frac{\mathcal{E}_{0,n}^{n+1}}{\text{im}(\delta_{n+1,0}^{n+1})}.$$

Then $\text{im}(\delta_{n+1,0}^{n+1}) = \mathcal{E}_{0,n}^{n+1}$ and we conclude that

$$(3.32) \quad \delta_{n+1,0}^{n+1} : \mathcal{E}_{n+1,0}^{n+1} \longrightarrow \mathcal{E}_{0,n}^{n+1} \text{ is surjective.}$$

As in the proof of Theorem 3.1 $\delta_{n+1,0}^{n+1}$ has the following domain and co-domain:

$$\delta_{n+1,0}^{n+1} : \mathcal{E}_{n+1,0}^2 \longrightarrow \mathcal{E}_{0,n}^2 .$$

Step 7. By the naturality of the LHS spectral sequence, we have the following commutative diagram of group homomorphisms :

$$(3.33) \quad \begin{array}{ccc} H_{n+1}(C_0, H_0(A, \prod_{\alpha \in I} (\mathbb{Z}B)_\alpha)) & \xrightarrow{\delta_{n+1,0}^{n+1}} & H_0(C_0, H_n(A, \prod_{\alpha \in I} (\mathbb{Z}B)_\alpha)) \\ \mu_1 \downarrow & & \downarrow \mu_2 \\ H_{n+1}(C, H_0(A, \prod_{\alpha \in I} (\mathbb{Z}B)_\alpha)) & \xrightarrow{\psi_{n+1,0}^{n+1}} & H_0(C, H_n(A, \prod_{\alpha \in I} (\mathbb{Z}B)_\alpha)) \end{array}$$

where μ_1 and μ_2 are induced by $\nu : C_0 \rightarrow C$ and $\psi_{n+1,0}^{n+1}$ is the differential of the LHS spectral sequence

$$H_p(C, H_q(A, \prod_{\alpha \in I} (\mathbb{Z}B)_\alpha)) \xrightarrow[p]{\Rightarrow} H_{p+q}(B, \prod_{\alpha \in I} (\mathbb{Z}B)_\alpha)$$

associated to the short exact sequence $A \rightarrow B \rightarrow C$. Set $V = H_n(A, \prod_{\alpha \in I} (\mathbb{Z}B)_\alpha)$.

Recall that $\nu : C_0 \rightarrow C$ is induced by θ and V is a left $\mathbb{Z}C_0$ -module via the homomorphism ν . Thus we have

$$H_0(C_0, V) \cong \frac{V}{\text{Aug}(\mathbb{Z}C_0)V} = \frac{V}{\text{Aug}(\mathbb{Z}(\text{im}(\nu)))V} \quad \text{and} \quad H_0(C, V) \cong \frac{V}{\text{Aug}(\mathbb{Z}C)V},$$

where Aug denotes the augmentation ideal of the appropriate group algebra. Then

$$\mu_2 : \frac{V}{\text{Aug}(\mathbb{Z}(\text{im}(\nu)))V} \longrightarrow \frac{V}{\text{Aug}(\mathbb{Z}C)V}$$

is the enlargement homomorphism, hence is surjective. By the commutative diagram (3.33), since $\delta_{n+1,0}^{n+1}$ and μ_2 are both surjective, we deduce that $\psi_{n+1,0}^{n+1}$ is surjective too and hence by Proposition 3.2 we have that B is of type FP_{n+1} . \square

Note that several results in [15] are deduced as corollaries of the technical [15, Prop. 4.3]. As we proved the homological version Theorem 3.4 of [15, Prop. 4.3] we will deduce in the following propositions that the homological versions of several results of [15] hold too. The proofs of the following results will use significantly Theorem 3.4 plus ideas from [15].

The following proposition implies Theorem B. As the statement of Proposition 3.5 shows, in the case when the second short exact sequence splits, there is no need to assume that Q is of type FP_{n+2} as in the Homological n -($n+1$)-($n+2$) Conjecture.

Proposition 3.5. *Let $n \geq 1$ be a natural number, $N_1 \rightarrow G_1 \rightarrow Q$ and $N_2 \rightarrow G_2 \rightarrow Q$ be short exact sequences with G_1 and G_2 groups of type FP_{n+1} such that N_1 is of type FP_n and the second sequence splits. Then the fibre P , associated to the above short exact sequences, is of homological type FP_{n+1} . In particular the Homological n -($n+1$)-($n+2$) Conjecture holds if the second sequence splits.*

Proof. As in the proof of [15, Cor, 4.6] there is a homomorphism $\phi : G_1 \rightarrow P$ whose restriction to N_1 is the identity map. Then we obtain the following exact diagram

$$\begin{array}{ccccc} N_1 & \hookrightarrow & G_1 & \xrightarrow{\pi_1} & Q \\ \text{id}_{N_1} \downarrow & & \downarrow \phi & & \downarrow \sigma_2 \\ N_1 & \hookrightarrow & P & \xrightarrow{p_2} & G_2 \end{array}$$

Finally by Teorema 3.4, P is of type FP_{n+1} . \square

The following corollary proves Theorem C. The second part of Corollary 3.6 will be used in the proof of Theorem 4.1.

Corollary 3.6. *If the Homological n -($n+1$)-($n+2$) Conjecture holds whenever G_2 is a finitely generated free group then it holds in general. If the Homological n -($n+1$)-($n+2$) Conjecture holds whenever G_2 is a finitely generated free group and Q is finitely presented then it holds if Q is finitely presented without restrictions on G_2 .*

Proof. We prove the first statement, the proof of the second statement is the same. Let $N_1 \hookrightarrow G_1 \xrightarrow{\pi_1} Q$ and $N_2 \hookrightarrow G_2 \xrightarrow{\pi_2} Q$ be short exact sequences of groups such that N_1 is of type FP_n , G_1 and G_2 are of type FP_{n+1} and Q is of type FP_{n+2} . Let P be the fibre product

$$P = \{(g_1, g_2) \in G_1 \times G_2 : \pi_1(g_1) = \pi_2(g_2)\}$$

and let $p : F \twoheadrightarrow G_2$ be an epimorphism, where F is a finitely generated free group. Consider the short exact sequences of groups $N_1 \hookrightarrow G_1 \xrightarrow{\pi_1} Q$ and $\ker(\pi_2 \circ p) \hookrightarrow F \xrightarrow{\pi_2 \circ p} Q$, and denote by P' the fibre product

$$P' = \{(g_1, f) \in G_1 \times F : \pi_1(g_1) = \pi_2 \circ p(f)\}.$$

By assumption, the homological n -($n+1$)-($n+2$) Conjecture holds whenever the middle group in the second exact sequence is finitely generated and free, so P' is of type FP_{n+1} . Consider the following commutative diagram, whose rows are short exact sequences

$$\begin{array}{ccccc} N_1 \times \mathbf{1} & \hookrightarrow & P' & \xrightarrow{p'_2} & F \\ \text{id}_{N_1} \times \mathbf{1} \downarrow & & \downarrow \text{id}_{G_1} \times p & & \downarrow p \\ N_1 \times \mathbf{1} & \hookrightarrow & P & \xrightarrow{p_2} & G_2 \end{array}$$

where $p'_2 : P' \twoheadrightarrow F$ is given by $p'_2(g_1, f) = f$, $p_2 : P \twoheadrightarrow G_2$ is defined by $p_2(g_1, g_2) = g_2$ and $\text{id}_{G_1} \times p : P' \rightarrow P$ is restriction of $\text{id}_{G_1} \times p : G_1 \times F \rightarrow G_1 \times G_2$. Then by Theorem 3.4 P is of type FP_{n+1} . \square

The following result is Theorem D from the introduction.

Theorem 3.7. *Let $n \geq 1$, $N_1 \hookrightarrow G_1 \xrightarrow{\pi_1} Q$ and $N_2 \hookrightarrow G_2 \xrightarrow{\pi_2} Q$ be short exact sequences of groups, where G_1, G_2 are of type FP_{n+1} , Q is virtually abelian, N_1 is of type FP_k and N_2 is of type FP_l for some $k, l \geq 0$ with $k + l \geq n$. Then the fibre product P of π_1 and π_2 is of type FP_{n+1} .*

Proof. We assume first that Q is abelian. Then $P \triangleleft (G_1 \times G_2)$ and $(G_1 \times G_2)/P$ is abelian. By Theorem 2.3, to complete the proof we have to show that for every character $\chi : G_1 \times G_2 \rightarrow \mathbb{R}$ with $\chi(P) = 0$, we have $[\chi] \in \Sigma^{n+1}(G_1 \times G_2, \mathbb{Z})$. Let χ be such a character. Observe that $G_1 \times G_2 = (G_1 \times \mathbf{1})P$ and $G_1 \times G_2 = P(\mathbf{1} \times G_2)$. Thus $\chi|_{(G_1 \times \mathbf{1})} \neq 0$, otherwise $\chi : (G_1 \times \mathbf{1})P \rightarrow \mathbb{R}$ would be the zero character. Similarly, $\chi|_{(\mathbf{1} \times G_2)} \neq 0$.

Since Q is abelian, $G_1/N_1, G_2/N_2$ are abelian. Note that $N_1 \cup N_2 \subseteq P$ and $\chi(P) = 0$ imply that $\chi|_{(G_1 \times \mathbf{1})}(N_1) = \mathbf{0}$ and $\chi|_{(\mathbf{1} \times G_2)}(N_2) = \mathbf{0}$. Let $k', l' \geq 0$ be such that $k' \leq k, l' \leq l$ and $k' + l' = n$. Since N_1 is of type FP_k and N_2 is of type FP_l , then N_1 is of type $FP_{k'}$ and N_2 is of type $FP_{l'}$. Thus, by Theorem 2.3, $[\chi|_{(G_1 \times \mathbf{1})}] \in \Sigma^{k'}(G_1, \mathbb{Z})$ and $[\chi|_{(\mathbf{1} \times G_2)}] \in \Sigma^{l'}(G_2, \mathbb{Z})$. By Theorem 2.4

$$[\chi] \in \Sigma^{k'+l'+1}(G_1 \times G_2, \mathbb{Z}) = \Sigma^{n+1}(G_1 \times G_2, \mathbb{Z}).$$

Now we consider the general case i.e. there is a normal abelian subgroup A of finite index in Q . Consider the short exact sequence of groups $N_1 \rightarrow \pi_1^{-1}(A) \xrightarrow{\pi_1} A$ and $N_2 \rightarrow \pi_2^{-1}(A) \xrightarrow{\pi_2} A$. Since $[G_1 : \pi_1^{-1}(A)] = [Q : A] < \infty$ and G_1 is of type FP_{n+1} , it follows that $\pi_1^{-1}(A)$ is of type FP_{n+1} . Similarly $\pi_2^{-1}(A)$ is of type FP_{n+1} . By the abelian case discussed above, the fibre product \tilde{P} of $\tilde{\pi}_1$ and $\tilde{\pi}_2$ is of type FP_{n+1} . Note that $\tilde{P} = P \cap (\pi_1^{-1}(A) \times \pi_2^{-1}(A))$, hence $[P : \tilde{P}] < \infty$. Since going up or down with a finite index does not change the homological type we deduce that P is of type FP_{n+1} . \square

We finish the section with the proof of the first part of Theorem F.

Theorem 3.8. *If $n \geq 2$ and the Homological $(n-1)$ - n -($n+1$) Conjecture holds whenever Q is virtually nilpotent then the Homological Virtual Surjection Theorem holds.*

Proof. The proof is similar to the proof of [15, Thm. 3.10], where instead of [15, Prop. 4.3] we apply Theorem 3.4 and we swap the numbers n and k in the proof of [15, Claim, Thm. 3.10]. We sketch the proof. Let $P \subseteq G_1 \times \dots \times G_k$ be as in the statement of the Homological Virtual Surjection Theorem. Then by substituting each G_i with a subgroup of finite index if necessary, we can assume that $P \subseteq G_1 \times \dots \times G_k$ is a subdirect product i.e. $p_i(P) = G_i$ for every $1 \leq i \leq k$. By [6, Prop. 3.2] or [15, Lemma 3.2] we obtain that $G_i/(P \cap G_i)$ is virtually nilpotent for every i .

Let $T = p_{1,\dots,k-1}(P)$, where $p_{1,\dots,k-1} : G_1 \times \dots \times G_k \rightarrow G_1 \times \dots \times G_{k-1}$ is the canonical projection and $N_{1,\dots,k-1} = P \cap T$. As in the proof of [15, Thm. 3.10] the fact that P virtually surjects on n factors implies that $N_{1,\dots,k-1} \subseteq G_1 \times \dots \times G_{k-1}$ virtually surjects on $n-1$ factors. If the Homological Virtual Surjection Theorem holds for smaller values of k , i.e. we use induction on k , then $N_{1,\dots,k-1}$ is of type FP_{n-1} . Furthermore $T \subseteq G_1 \times \dots \times G_{k-1}$ virtually surjects on n -tuples if $n \leq k-1$, so by induction on k the group T is of type FP_n . If $n \geq k$ we get that $n = k$ and in this case the Homological Virtual Surjection Theorem obviously holds.

As in the proof of [15, Thm. 3.10] and using [15, Lemma 2.3] for the subdirect product $P \subseteq T \times G_k$, we deduce that P is the fibre product associated to the short exact sequences $N_{1,\dots,k-1} \rightarrow T \rightarrow Q$ and $N_k \rightarrow G_k \rightarrow Q$, where $N_k = P \cap G_k$. Thus $Q \simeq G_k/N_k$ is virtually nilpotent. Then we can apply the Homological $(n-1)$ - n -($n+1$) Conjecture since Q is virtually nilpotent and obtain that P is of type FP_n . \square

4. THE HOMOLOGICAL 1-2-3 CONJECTURE FOR FINITELY PRESENTED Q

In this section we prove Theorem E.

Theorem 4.1. *The Homological 1-2-3 Conjecture holds if in addition Q is finitely presented.*

Proof. Let $A \hookrightarrow G_1 \xrightarrow{\pi_1} Q$ and $B \hookrightarrow G_2 \xrightarrow{\pi_2} Q$ be short exact sequences of groups with A finitely generated, G_1 and G_2 of type FP_2 and Q is FP_3 and finitely presented. Denote by P the associated fibre product. We aim to show that P is of type FP_2 . By Corollary 3.6 we can assume that G_2 is a free group F with a finite free basis X . Let

$$\langle X \mid \tilde{R} \rangle, \text{ where } \tilde{R} = \{r_i(\underline{x})\}_{i \in I_0},$$

be a finite presentation of Q . Then there is a presentation of the group G_1

$$G_1 = \langle X \cup A_0 \mid R_1 \cup R_2 \cup R_3 \rangle,$$

where $A_0 = \{a_1, a_2, \dots, a_k\}$ is a finite set of generators of A and

$$R_1 = \{r_i(\underline{x})w_i(\underline{a})^{-1}\}_{i \in I_0}, R_2 = \{a_j^x v_{j,x}(\underline{a})^{-1}\}_{1 \leq j \leq k, x \in X \cup X^{-1}} \text{ and } R_3 = \{z_j(\underline{a})\}_{j \in J}$$

for some possibly infinite index set J and $w_i(\underline{a}), v_{j,x}(\underline{a}), z_j(\underline{a})$ are elements of the free group with a free basis A_0 .

Let R be the normal subgroup of the free group $F(X \cup A_0)$ with a free basis $X \cup A_0$ generated as a normal subgroup by $R_1 \cup R_2 \cup R_3$. Then there is a short exact sequence of groups

$$R \twoheadrightarrow F(X \cup A_0) \twoheadrightarrow G_1.$$

Since G_1 is FP_2 we obtain that $R/[R, R]$ is finitely generated as $\mathbb{Z}G_1$ -module via conjugation. Hence there is a finite subset J_0 of J such that

$$(4.1) \quad R = R_0[R, R],$$

where R_0 is the normal closure of $R_1 \cup R_2 \cup R_{3,0}$ in the free group $F(X \cup A_0)$ and $R_{3,0} = \{z_j(\underline{a})\}_{j \in J_0} \subseteq R_3$.

Consider the group $\tilde{G}_1 = F(X \cup A_0)/R_0$. Then there is a natural projection

$$\pi : \tilde{G}_1 = F(X \cup A_0)/R_0 \rightarrow G_1 = F(X \cup A_0)/R$$

with kernel $S = R/R_0$. By (4.1) we have

$$(4.2) \quad S = [S, S].$$

Since $R_1 \cup R_2$ are relations in \tilde{G}_1 we deduce that the subgroup \tilde{A}_1 of \tilde{G}_1 generated by the elements of A_0 is a normal subgroup of \tilde{G}_1 such that $\tilde{G}_1/\tilde{A}_1 \simeq Q$. Thus there is a short exact sequence of groups

$$(4.3) \quad \tilde{A}_1 \twoheadrightarrow \tilde{G}_1 \xrightarrow{\tilde{\pi}_1} Q$$

with both \tilde{G}_1 and Q finitely presented. Denote by \tilde{P}_1 the fibre of the short exact sequences (4.3) and $B \twoheadrightarrow F \xrightarrow{\pi_2} Q$, thus

$$\tilde{P}_1 = \{(h_1, h_2) \in \tilde{G}_1 \times F \mid \tilde{\pi}_1(h_1) = \pi_2(h_2)\}.$$

Since Q is FP_3 and is finitely presented, it is F_3 . Then the 1-2-3 Theorem from [6] implies that \tilde{P}_1 is finitely presented.

Recall that P is the fibre of the original short exact sequences $A \rightarrowtail G_1 \xrightarrow{\pi_1} Q$ and $B \rightarrowtail F \xrightarrow{\pi_2} Q$, i.e.

$$P = \{(h_1, h_2) \in G_1 \times F \mid \pi_1(h_1) = \pi_2(h_2)\}.$$

The map $\pi \times id_F : \tilde{G}_1 \times F \rightarrow G_1 \times F$ induces an epimorphism

$$\mu : \tilde{P}_1 \twoheadrightarrow P$$

with kernel $\ker(\mu) = S$, where $S = \ker(\pi)$. Then $\tilde{P}_1 = F(X \cup A_0)/S_1$ for some normal subgroup S_1 of $F(X \cup A_0)$ and $P = F(X \cup A_0)/S_2$ for some normal subgroup S_2 of $F(X \cup A_0)$ such that $S_1 \subseteq S_2$ and $S_2/S_1 = S = [S, S]$. Hence

$$S_2 = S_1[S_2, S_2],$$

so $S_2/[S_2, S_2]$ is an epimorphic image of the abelianization $S_1/[S_1, S_1]$. Since \tilde{P}_1 is finitely presented, it is of type FP_2 . Then we deduce that $S_1/[S_1, S_1]$ is finitely generated as $\mathbb{Z}\tilde{P}_1$ -module via conjugation. Thus its epimorphic image $S_2/[S_2, S_2]$ is finitely generated as $\mathbb{Z}P$ -module via conjugation, hence P is of type FP_2 as claimed. \square

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STATE UNIVERSITY OF CAMPINAS (UNICAMP), SP, BRAZIL, EMAIL : DESI@IME.UNICAMP.BR
 , TECHNICAL STATE UNIVERSITY OF PARANÁ (UTFPR), PR, BRAZIL, EMAIL : FRANCISMARFER-
 REIRALIMA@GMAIL.COM,